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More Diffy Q techniques: Separable Differential Equations

Goal: Be able to solve differential equations of the form $\frac{dy}{dx} = f(x) \cdot g(y)$

Separable Differential Equations	Not Separable Differential Equations
$\frac{dy}{dx} = -\frac{x}{y} = (-x) \cdot \left(\frac{1}{y}\right)$	$\frac{dy}{dx} = x + y$
$\frac{dx}{dt} = xy$	$\frac{dx}{dt} = \frac{t+x}{xt^2}$
$\frac{dy}{dx} = yx^2 + yx + y = (x^2+x+1) \cdot y$	

A short Tangent: Differentials

$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$ except change is infinitesimal.

dy = infinitesimal change in y

dx = infinitesimal change in x

When we take derivatives and antiderivatives, the dy, dx notation refers to the same thing:

differentials $\left[\begin{array}{l} dy \\ dx \end{array}\right.$, $\int f(x)\,\underbrace{dx}_{\text{differential}}$ If $\frac{dy}{dx} = f'(x)$

$dy = f'(x)dx$

we will have a more geometric notion of antiderivatives and differentials next week

Back to Sep. Diffy Q's

Ex: (1) $\frac{dy}{dx} = -\frac{x}{y}$

*let's put y's and dy's on one side of equation and x's and dx's on the other side. this is the "separation"

$$y \cdot dy = -x \cdot dx$$

(2) $\frac{dy}{dx} = xy$

$\frac{dy}{y} = x \cdot dx$

to solve (1), (2) we take the antiderivative of both sides

$$\int y \cdot dy = \int -x\,dx$$

$$\frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C \quad (C = C_2 - C_1)$$

Now solve for y

$$y^2 = -x^2 + 2C$$

$$y = \pm\sqrt{\underbrace{2C}_{\text{replace w/ another letter to look cleaner (doesn't actually matter)}} - x^2}$$

$$y = \pm\sqrt{A - x^2} \quad (A = 2C)$$

(2) $$\int \frac{dy}{y} = \int x \cdot dx$$

$$\int \left(\frac{1}{y}\right) dy = \frac{x^2}{2} + C_2$$

$$\ln|y| + C_1 = \frac{x^2}{2} + C_2$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C}$$

$$y = e^{x^2/2} \cdot \underbrace{e^C}_{\text{a constant}} \quad (\text{let } K = e^C)$$

$$y = Ke^{x^2/2}$$

Ex: (3) $\frac{dy}{dx} = 3-2y$ this is separable. divide both sides by $3-2y$

$$\int \underbrace{\frac{1}{3-2y}}_{\text{Need to guess a bit to get this antiderivative}} \cdot dy = \int dx$$

$\ln(3-2y)$ has derivative

$$\frac{1}{3-2y} \cdot -2$$

so $\frac{\ln(3-2y)}{-2}$

has derivative

$$\frac{1}{3-2y}$$

$$\frac{\ln(3-2y)}{-2} = x + C$$

$$\ln(3-2y) = -2x - 2C$$

$$3-2y = e^{-2x-2C}$$

$$-2y = e^{-2x}e^{-2C} - 3 \quad (k = e^{-2C})$$

$$y = k\,\frac{e^{-2x}}{-2} + \frac{3}{2}$$

(4) $\frac{dy}{dx} = \frac{1}{y\cos^2 x}$

$$\int \frac{1}{y} \cdot dy = \int \underbrace{\frac{1}{\cos^2 x}}_{\sec^2 x} dx$$

$$\ln y = \tan x + C$$

$$y = e^{\tan x + C}$$

$$y = ke^{\tan x}$$